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Curvature-induced frustration in the *XY* model on hyperbolic surfaces

Seung Ki Baek,[1,*] Hiroyuki Shima,[2,†] and Beom Jun Kim[3,4,‡]
[1]*Department of Physics, Umeå University, 901 87 Umeå, Sweden*
[2]*Department of Applied Physics, Graduate School of Engineering, Hokkaido University, Sapporo 060-8628, Japan*
[3]*BK21 Physics Research Division and Department of Energy Science, Sungkyunkwan University, Suwon 440-746, Korea*
[4]*Department of Computational Biology, School of Computer Science and Communication, Royal Institute of Technology, 100 44 Stockholm, Sweden*

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We study low-temperature properties of the *XY* spin model on a negatively curved surface. Geometric curvature of the surface gives rise to frustration in local spin configuration, which results in the formation of high-energy spin clusters scattered over the system. Asymptotic behavior of the spin-glass susceptibility suggests a zero-temperature glass transition, which is attributed to multiple optimal configurations of spin clusters due to nonzero surface curvature of the system. It implies that a constant ferromagnetic spin interaction on a regular lattice can exhibit glasslike behavior without possessing any disorder if the lattice is put on top of a negatively curved space such as a hyperbolic surface.

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The minimum energy principle that a physical system obeys may be broken by applying external conditions. One such condition, which has profound consequences in the ergodicity of the system, is geometric confinement to a curved surface. A typical example is an ensemble of classical electrons confined to a spherical surface [1]. It exhibits so many quasistable configurations that a unique ground state is hardly observed within a feasible time scale [2]. These long-lived states result from the incompatibility between crystalline order and surface curvature. In fact, partial disorders such as defects and dislocations are necessary in order for local crystalline order to propagate through a curved surface, wherein various possible configurations of disorders cause many frustrated states. Such a curvature-induced frustration has also been observed on more general geometries [2,3].

Another important consequence of nonzero surface curvature is a breakdown of an *orientational* order. When interacting constituents on a curved surface have orientational degrees of freedom, they can no longer show perfect orientational order. The loss of perfect orientational order is due to the noncommutative property of parallel transport of vectors [4]. On a curved surface, parallel transport of a vector along a closed loop does not maintain its direction but yields a rotation after the round trip. This makes it impossible for all vectors to orient the same direction yielding multiple frustrated states at low temperatures even if the system does not possess any disorder [5]. These facts imply a novel class of orientational glasses free from any kind of disorder, which is in contrast with ordinary spin glasses dominated by certain disorder (see, e.g., Refs. [6,7] for attempts at glass formation in absence of intrinsic disorder). Understanding the nature of such curvature-induced glass transition, if it exists, should be crucial from viewpoints of statistical physics and soft material sciences. Particularly in the latter field, systems with curved or fluctuating geometries are accessible to synthesize [8], although the interplay between the geometry and thermodynamic properties of allowed ordered states still remains to be explored [9].

There are two well-established approaches to analyze the orientational order of a physical system: discrete lattice simulations and continuum limit approximations. The former approach is preferable to study actual evolutions of orientational order. On curved surfaces, however, one cannot construct regular lattices with congruent polygons, in general. Hence, the resulting lattice usually involves structural defects as mentioned in the first paragraph. These defects may give additional contributions to the allowed orientational configuration, thus should be removed when we are to extract purely curvature effects. This can be achieved by employing a surface having constant negative Gaussian curvature called a hyperbolic surface [10]. This curved surface enables to construct a wide range of regular lattices (called a hyperbolic lattice) on it, serving as a platform to address the issue.

In this Rapid Communication, we consider effects of curvature-induced frustration on the orientational order in the *XY* spin model defined on a hyperbolic lattice. Monte Carlo (MC) simulations are used to evaluate the spin configurations at low temperatures revealing the formation of high-energy spin clusters distributed in the system. We propose that these configurations are multiply degenerate due to curvature-induced frustration, which may lead to a zero-temperature glass transition as supported by calculating the spin-glass susceptibility.

The hyperbolic lattice we have used is depicted in Fig. 1(a) in terms of the Poincaré disk representation [11]. The lattice consists of equilateral heptagons in the metric of the hyperbolic surface so that we call it a regular heptagonal lattice. Seven vertices in the central heptagon make the first layer denoted by $l=1$, which are surrounded in a concentric way by the second ($l=2$) and third ($l=3$) layers. The number of vertices $N(l)$ increases exponentially with l, making this lattice infinite dimensional. This infinite-dimensional property has been found to yield various nontrivial thermody-

*garuda@tp.umu.se
†shima@eng.hokudai.ac.jp
‡beomjun@skku.edu

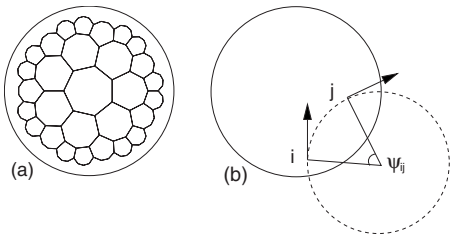

FIG. 1. (a) A heptagonal lattice with $l=3$ in the Poincaré disk. The circumference of the disk represents the points at infinity. (b) Parallelism between two spins in the Poincaré disk. Parallel transport of an arrow from ith to jth site along the geodesic curve (dotted) rotates the arrow by ψ_{ij}.

namic properties of physical systems defined on this lattice [12]. The relevance to the mean-field behavior observed in small-world networks has also been discussed [13].

Let us imagine two XY spins i and j confined in the surface and a geodesic between them. On the analogy of the continuum limit, we suppose that i's phase is experienced by j after parallel transported to j's position along the geodesic. The Hamiltonian is then given by

$$\mathcal{H} = -J\sum_{\langle ij\rangle} \cos(\phi_i - \phi_j - \psi_{ij}), \tag{1}$$

where J is the coupling constant, the sum is over all nearest pairs in the system, and ϕ_i represents the spin variable at the ith site. Nonzero surface curvature manifests in the presence of the additional angle ψ_{ij}. The angle ψ_{ij} describes the amount that i's phase acquires after the parallel transport. In the language of differential geometry, this additional angle stems from the affine connection [4] between two neighboring sites along the geodesic line. Figure 1(b) illustrates how to actually determine ψ_{ij}: we obtain positions of i and j on the Poincaré disk by hyperbolic tessellation [10]. If they lie on a line through the origin of the Poincaré disk, the geodesic is represented by a straight line yielding $\psi_{ij}=0$. Otherwise, the geodesic appears as an arc of a circle that meets the circumference of the Poincaré disk at a right angle. The positions of i and j together with the origin of this circle make an angle, which determines ψ_{ij}.

The spins at sites i and j are parallel when $\phi_i-\phi_j-\psi_{ij}=0$. (Note that $\psi_{ij}=-\psi_{ji}$ to keep the spin-spin interaction symmetric.) Accordingly, parallel transport of the ith spin along seven edges of a heptagon alters the value of ϕ_i causing frustration in local spin configuration. It should be emphasized that the local frustration in our system has no relation with any quenched disorder and originates purely from the intrinsic geometry of the surface. Such a disorder-free frustration is in contrast with disorder-driven frustrations in random gauge XY models [14]. It is also notable to see how this differs from such frustrations as observed in the triangular antiferromagnetic Ising (AFI) model [7]: while the hexagonal symmetry in AFI allows the presence of loose spins as well as the fully ordered configuration, none of these can be found in our system. We point out that the sum of ψ_{ij} around each heptagon is conserved as $\Sigma_P\psi_{ij}=2\pi f$ with $f=-1/6$; here Σ_P represents the summation around a plaquette in a counterclockwise direction and f characterizes the strength of frustration. The quantitative invariance of f for all constituent heptagons is analogous to that in the uniformly frustrated XY model on a flat plane [15]. The latter model describes a superconducting film penetrated by external magnetic flux [16] and phase ordering in it is still under active investigations [17]. It is known that the uniformly frustrated XY model exhibits glassy behavior on a square lattice when f is irrational [15,18]. We should notice, nevertheless, that a direct analogy between the two systems is hindered by topological differences between planar and hyperbolic lattices. In fact, the hyperbolic lattices show glassy behavior for a rational f as demonstrated below.

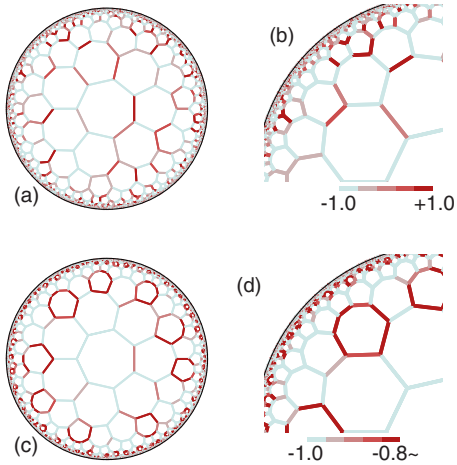

FIG. 2. (Color online) Snapshots of spin configurations with $l=7$, where a darker line represents a higher-energy bond. (a) A high-temperature regime with $T=1.00$ and (b) a closer look at the surface. (c) A low-temperature regime with $T\approx 0.06$ and (d) its magnified image, where clustering structures are clearly visible. The bond energy and temperature T are in units of J and J/k_B, respectively.

To equilibrate the system described by Eq. (1), we employ MC simulations incorporated with the parallel tempering (PT) method [19]. It is a numerical technique devised to take averages efficiently over the state space. In this method, MC calculations are carried out simultaneously at different temperatures, which significantly saves computational cost for equilibrating large-scale frustrated systems. Suppose that we have spin configurations $\{\phi_i\}_{T_k}$ and $\{\phi_i\}_{T_{k+1}}$ running at temperatures $T=T_k$ and T_{k+1}, respectively. Without loss of generality, we may set $T_k>T_{k+1}$. Performing the standard Metropolis algorithm on the configurations, we regularly check their energies $E[\{\phi_i\}_{T_k}]$ and $E[\{\phi_i\}_{T_{k+1}}]$ and exchange these configurations with a probability of $\min[1, e^{-(E[\{\phi_i\}_{T_k}]-E[\{\phi_i\}_{T_{k+1}}])(1/T_{k+1}-1/T_k)}]$. That is, when $\{\phi_i\}_{T_{k+1}}$ is trapped in a local energy minimum, PT makes it probe a wider region of the state space by passing it to a higher temperature, T_k. If it finds a state with a sufficiently low energy, the vicinity can be checked in more detail by lowering its temperature again.

Figure 2 plots the resulting spin configurations at high and low temperatures. In the former two figures, there appears no regularity in the distribution of high-energy bonds (denoted by bonds with darker tone). On the contrary, at low T, high-

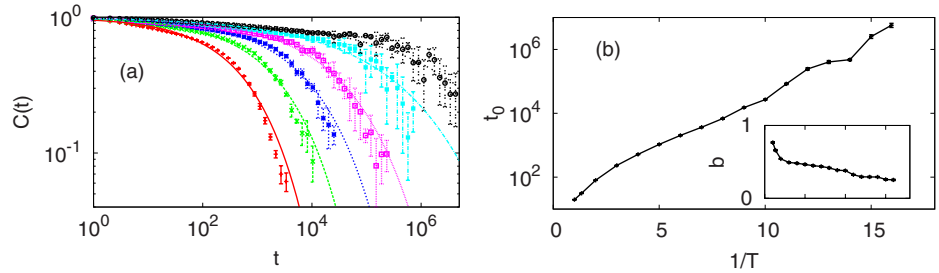

FIG. 3. (Color online) (a) Autocorrelations of spin configurations, given in Eq. (2), together with fitting results with a stretched-exponential form. The results are obtained for $T=0.250$, 0.167, 0.125, 0.100, 0.083, and 0.067, from bottom to top. (b) Relaxation time t_0 as a function of inverse temperature from the stretched-exponential fit. Inset: b as a function of $1/T$ from the same fit. It approaches unity at high T leading to a simple exponential form.

energy bonds are merged into localized regions forming clusters. It implies that the formation of such high-energy clusters is energetically favorable: even though frustration cannot be removed, they may be shifted and confined into small regions, lowering the energy in other regions instead. Furthermore, we have observed that the locations of such high-energy clusters remain floating by thermal fluctuations. Thereby the distribution of high- and low-energy bonds is not unique making the ground state degenerate. Indeed, we find that each high-energy cluster in Fig. 2 appears as a topological defect, i.e., a negative vortex to reduce the free-energy cost induced by the curvature [20]. It is noteworthy that those plots show a similarity to the results in Ref. [21], which reports that defects of a glass-forming liquid in a negatively curved space are concentrated in local regions at low T. The origin of negative vortices differs inherently from thermal dissociation of a vortex-antivortex pair on a hyperbolic surface whose properties were considered in Ref. [22]. Still, we note that if the interaction between vortices becomes short ranged due to curvature [22], this effect may be relevant in exhibiting glassy features.

The glassy behavior is quantified by the relaxation time t_0 for the system to evolve from the configurations obtained in PT. To this aim, we calculate autocorrelation

$$C(t) = \frac{1}{N\tau}\sum_{t'=1}^{\tau}\left|\sum_{j=1}^{N} e^{i[\phi_j(t+t')-\phi_j(t')]}\right|, \qquad (2)$$

with the standard Metropolis algorithm for $\tau=10^4$ MC time steps. We extract t_0 by fitting $C(t)$ with a stretched-exponential function, $C(t)\propto\exp[-(t/t_0)^b]$ [23], characterizing very slow relaxation in glasses. Figures 3(a) and 3(b) display the measurements for $l=7$ and fitting results. The relaxation behavior is described quite well by the simple Arrhenius form, $t_0\propto\exp(\Delta/T)$, in which Δ indicates the activation energy. It is noteworthy that the persistent linearity of $\log t_0$ with respect to $1/T$ shown in Fig. 3(b) (i.e., T-independent activation energy Δ) is commonly observed in supercooled liquids such as v-SiO_2 and v-GeO_2, in which spatiotemporal fluctuation of local configuration leads to critical slowing down [24].

The presence of a glass transition as well as transition temperature can be probed by observing the divergence of the spin-glass susceptibility designated by χ_{SG}. In performing PT to calculate χ_{SG}, we prepare two replicas μ and ν for each T and N. Then we obtain the spin-glass susceptibility $\chi_{SG}=NS(\tau;t_0')/\tau$, where

$$S(\tau;t_0') \equiv \sum_{t'=1}^{\tau}\left|\frac{1}{N}\sum_{j=1}^{N} e^{i[\phi_j^{\mu}(t_0'+t')-\phi_j^{\nu}(t_0'+t')]}\right|^2, \qquad (3)$$

with the measurement time τ after some equilibration time t_0' [25]. Figure 4(a) gives the T dependence of $\chi_{SG}(T,N)$ with varying N's. Our simulation code can properly handle the numerical precision up to $l=7$, and the cases for $l<5$ are excluded due to finite-size effects. We see that χ_{SG} begins to increase at low T. In order to evaluate the glass transition temperature T_g, χ_{SG} is plotted against N on a log-log scale in Fig. 4(b). Strikingly, all the results of χ_{SG} give no sign of attaining a power law of N at any finite T we have tried. This implies that a true singularity lies only at $T=0$. We thus conclude that $T_g=0$; i.e., this system undergoes a zero-temperature glass transition. More detailed finite-size analysis remains to be done largely due to lack of available system sizes. One interesting point is that a hyperbolic lattice does

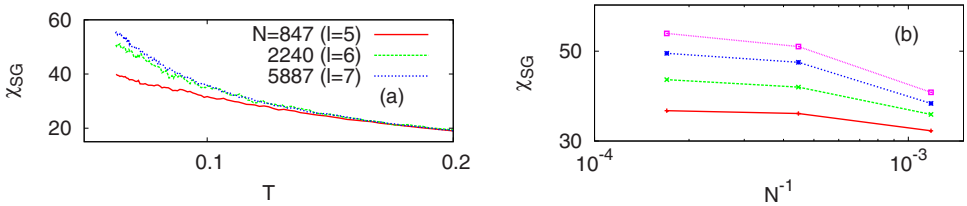

FIG. 4. (Color online) (a) Spin-glass susceptibility χ_{SG} as a function of T. (b) χ_{SG} plotted against N^{-1} at $T=0.100$, 0.083, 0.071, and 0.063, from bottom to top. It does not exhibit a power law with respect to N for any finite T.

not possess scale invariance as it has its own length scale, i.e., the radius of curvature [26], which may modify the nature of the singularity.

In summary, we have demonstrated an apparent zero-temperature orientational glass transition in the *XY* spin model on a negatively curved surface. MC simulations revealed that lowering T makes high-energy bonds gather to form clusters, which remain floating with very long relaxation times. The long lifetime fluctuation in cluster distribution follows an Arrhenius-type relaxation at low T and the singularity of spin-glass susceptibility is expected to arise only at $T_g=0$. These observations are consequences of geometry of the surface, where curvature-induced frustration in local (and thus global) spin configurations yields ground-state degeneracy.

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